

# The influence of the chameleon field potential on transition frequencies of gravitationally bound quantum states of ultra-cold neutrons

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We calculate the chameleon field potential for ultracold neutrons, bouncing on top of one or between two neutron mirrors in the gravitational field of the Earth. For the resulting non-linear equations of motion we give approximate analytical solutions and compare them with exact numerical ones for which we propose the analytical fit. The obtained solutions may be used for the quantitative analysis of contributions of a chameleon field to the transition frequencies of quantum states of ultra-cold neutrons bound in the gravitational field of the Earth.

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## I. INTRODUCTION

A chameleon field has been suggested to drive the current phase of cosmic acceleration for a large class of scalar potentials. The properties depend on the density of a matter to which it is immersed. Because of this sensitivity on the environment, the field was called chameleon, and it can couple directly to baryons with gravitational strength on Earth but it would be essentially massless on solar system scales [1, 2].

An interaction of a chameleon field to an environment with a mass density  $\rho$  and its self-interaction are described by the effective potential  $V_{\text{eff}}(\phi)$  [3]

$$V_{\text{eff}}(\phi) = V(\phi) + \rho e^{\beta\phi/M}, \quad (1)$$

where  $\beta$  is a coupling constant and  $M = 1/\sqrt{8\pi G_N} = 2.435 \times 10^{18} \text{ GeV}$  is the Planck mass.

As has been pointed out by Ref. [3], ultra-cold neutrons, bouncing in the gravitational field of the Earth above a mirror can be a good laboratory for testing the existence of a chameleon field. There has been found a solution of the equations of motion and limits for the coupling constant  $\beta$  can be estimated from the contribution of a chameleon field to the transition frequencies of the quantum gravitational states of ultra-cold neutrons bouncing in the gravitational field of the Earth. Such resonant transitions between quantum states of a neutron in the gravity potential have been measured by the qBounce Collaboration [5]. In this experiment, ultra-cold neutrons form gravitationally bound quantum states between a neutron mirror on bottom and another neutron mirror on top with a relative distance  $d$ . The upper neutron mirror has a rough surface and thereby filters out higher, unwanted states. The whole system may be vibrated in vertical direction with frequency  $\omega$  and amplitude  $a$ . Behind this system, the neutron transmission

is measured. It shows a significant reduction in count rate, if the vibration frequency corresponds to the energy difference between two eigenstates. With this method, the transition frequencies have been determined, which tests Newton's Law at short distances and is therefore sensitive to any deviation like chameleon fields. So far, such chameleon fields have only been calculated in the infinitely large spatial region above one mirror [3]. For the experiment presented in [5], it is necessary to use solutions of a chameleon field localized between two mirrors for the estimate of the coupling constant  $\beta$ .

As we show below the appearance of an additional mirror complicates the problem of the calculation of a chameleon field substantially. As a result we propose the analysis of all possible solutions of the problem under consideration.

The paper is organized as follows: In section II, the chameleon field for ultra-cold neutrons, bouncing in vacuum in the gravitational field of the Earth above a horizontal mirror, is calculated. We show that the solution obtained in [3] is exact for this problem. In sections III, the calculation of the chameleon field is adapted to the experimental setup used in [5]. For this purpose, we consider a symmetric geometry, where the horizontal mirrors occupy the spatial regions  $z^2 > \frac{d^2}{4}$ , whereas ultra-cold neutrons bounce in the spatial region  $z^2 \leq \frac{d^2}{4}$ . We obtain the solutions for a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$ . In section IV we solve the problem of a chameleon field between two mirrors numerically without any approximation and give an analytical fit of these solutions for arbitrary power  $n$ .

## II. THE CHAMELEON FIELD COUPLED TO ULTRA-COLD NEUTRONS BOUNCING ABOVE A MIRROR

According to Ref.[3, 4], the potential  $V(\phi)$  takes the form:

$$V(\phi) = \Lambda^4 + \frac{\Lambda^{4+n}}{\phi^n}. \quad (2)$$

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The scale  $\Lambda$  is chosen to be equal to  $\Lambda = 2.4(1) \times 10^{-12}$  GeV [3]. Following [3] and keeping only the linear term in the expansion of the exponential  $e^{\beta\phi/M}$  in powers of  $\phi$  one can show that the effective potential has a minimum

$$V'_{\text{eff}}(\phi_{\min}) = V'(\phi_{\min}) + \rho \frac{\beta}{M} = 0, \quad (3)$$

where  $V'_{\text{eff}}(\phi)$  is a derivative with respect to  $\phi$ , at

$$\phi_{\min} = \Lambda \left( \frac{nM\Lambda^3}{\beta\rho} \right)^{1/(n+1)}. \quad (4)$$

Quanta of a chameleon field are massive with mass defined by the second derivative of the potential  $V(\phi)$  [3]. A chameleon field satisfies an equation of motion [3]:

$$\square\phi = -V'_{\text{eff}}(\phi) = -V'(\phi) + V'(\phi_{\min}). \quad (5)$$

As analysed by Brax *et al.* [3], a neutron mirror with mass density  $\rho_m$  being parallel to the  $(x, y)$ -plane at  $z = 0$ , occupies a spatial region  $z \leq 0$ , whereas ultra-cold neutrons bounce above in the gravity field in the spatial region  $z \geq 0$  with vacuum density  $\rho_v \simeq 0$ .

For such a geometry a chameleon field depends on the spatial variable  $z$  only and the equation of motion takes the form

$$\frac{d^2\phi}{dz^2} = V'(\phi) - V'(\phi_{\min}), \quad (6)$$

where  $\phi_{\min} = \phi_m$  and  $\phi_{\min} = \phi_v$  for spatial regions  $z \leq 0$  and  $z \geq 0$ , respectively. In our analysis of a chameleon field for a system mirror-vacuum we follow [3], but – in view of the following sections – locate the mirror at  $z = -\frac{d}{2}$ . In this case the mirror occupies the spatial region  $z \leq -\frac{d}{2}$ , whereas vacuum fills up the region  $z \geq -\frac{d}{2}$ .

After the first integration the equation of motion Eq. (6) reduces to the form

$$\frac{1}{2} \left( \frac{d\phi}{dz} \right)^2 = C + V(\phi) - \phi V'(\phi_{\min}), \quad (7)$$

where  $C$  is a constant of integration. This constant we obtain from the asymptotic conditions. This constant we obtain from the asymptotic conditions assuming that a chameleon field tends to its minimum and a vanishing derivative at  $z \rightarrow \pm\infty$  [3]:

$$\lim_{z \rightarrow \pm\infty} \phi(z) = \phi_m \quad \lim_{z \rightarrow \pm\infty} \frac{d\phi(z)}{dz} = 0. \quad (8)$$

These conditions result in

$$C = -V(\phi_{\min}) + \phi_{\min} V'(\phi_{\min}). \quad (9)$$

Substituting Eq.(9) into Eq.(7) we obtain

$$\frac{1}{2} \left( \frac{d\phi}{dz} \right)^2 = V(\phi) - V(\phi_{\min}) - (\phi - \phi_{\min}) V'(\phi_{\min}). \quad (10)$$

Thus, in spatial regions  $z \leq -\frac{d}{2}$  and  $z \geq -\frac{d}{2}$  a chameleon field should obey the equations

$$\begin{aligned} \frac{1}{2} \left( \frac{d\phi}{dz} \right)^2 &= V(\phi) - V(\phi_m) - (\phi - \phi_m) V'(\phi_m) = \\ &= \frac{\Lambda^{4+n}}{\phi^n} \left( 1 - (n+1) \frac{\phi^n}{\phi_m^n} + n \frac{\phi^{n+1}}{\phi_m^{n+1}} \right) \end{aligned} \quad (11)$$

and

$$\begin{aligned} \frac{1}{2} \left( \frac{d\phi}{dz} \right)^2 &= V(\phi) - V(\phi_v) - (\phi - \phi_v) V'(\phi_v) = \\ &= \frac{\Lambda^{4+n}}{\phi^n} \left( 1 - (n+1) \frac{\phi^n}{\phi_v^n} + n \frac{\phi^{n+1}}{\phi_v^{n+1}} \right), \end{aligned} \quad (12)$$

respectively, where we have used the exact shape of the potential  $V(\phi)$ , given by Eq.(2), and  $\phi_m$  and  $\phi_v$  are given by Eq.(4) for  $\rho = \rho_m$  and  $\rho = \rho_v$ , respectively.

The solutions of Eq.(11) and Eq.(12) should obey the boundary conditions

$$\begin{aligned} \phi(z) \Big|_{z \rightarrow -\frac{d}{2}-} &= \phi(z) \Big|_{z \rightarrow -\frac{d}{2}+} = \phi_d, \\ \frac{d\phi(z)}{dz} \Big|_{z \rightarrow -\frac{d}{2}-} &= \frac{d\phi(z)}{dz} \Big|_{z \rightarrow -\frac{d}{2}+}, \end{aligned} \quad (13)$$

where  $\phi_d = \phi(-d/2)$ . Since ultra-cold neutrons are in the spatial region  $z \geq -\frac{d}{2}$ , we have to search a solution of Eq.(12) only. As regards Eq.(11), it may be used only for the boundary conditions.

As a vacuum density  $\rho_v \simeq 0$  leads to  $\phi_v \rightarrow \infty$ , we take the r.h.s. of Eq.(12) in the limit  $\phi_v \rightarrow \infty$ . This reduces Eq.(12) to the form

$$\frac{1}{2} \left( \frac{d\phi}{dz} \right)^2 = \frac{\Lambda^{4+n}}{\phi^n}. \quad (14)$$

Since a chameleon field grows with  $z \rightarrow \infty$ , in the spatial region  $z \geq -\frac{d}{2}$  a derivative of a chameleon field is always positive. This gives a differential equation

$$\sqrt{2} \Lambda dz = \left( \frac{\phi}{\Lambda} \right)^{\frac{n}{2}} \frac{d\phi}{\Lambda}. \quad (15)$$

The solution of this equation is given by the integral

$$\sqrt{2} \Lambda \left( \frac{d}{2} + z \right) = \int_{\phi_d}^{\phi} \left( \frac{\phi}{\Lambda} \right)^{\frac{n}{2}} \frac{d\phi}{\Lambda}. \quad (16)$$

Integrating over  $\phi$  and solving the obtained expression with respect to  $\phi(z)$  we obtain

$$\phi(z) = \phi_d \left( 1 + \frac{n+2}{\sqrt{2n(n+1)}} \frac{1}{\lambda_d} \left( \frac{d}{2} + z \right) \right)^{\frac{2}{n+2}}, \quad (17)$$

where  $1/\lambda_d = \Lambda \sqrt{n(n+1)} (\Lambda/\phi_d)^{n+2}$ . At  $z = -\frac{d}{2}$  from a boundary condition on the derivatives of a chameleon field

$$\frac{\Lambda^{4+n}}{\phi_d^n} \left( 1 - (n+1) \frac{\phi_d^n}{\phi_m^n} + n \frac{\phi_d^{n+1}}{\phi_m^{n+1}} \right) = \frac{\Lambda^{4+n}}{\phi_d^n} \quad (18)$$

we define  $\phi_d$  in terms of  $\phi_m$

$$\phi_d = \frac{n+1}{n} \phi_m. \quad (19)$$

This relation is valid for arbitrary value of the coupling constant  $\beta$ . Since the solution Eq.(17) obeys the asymptotic conditions Eq.(8) at  $z \rightarrow \infty$  and the boundary conditions Eq.(13), it defines a chameleon field in the spatial region  $z \geq -\frac{d}{2}$ .

It is obvious that if a mirror with a density  $\rho_m$  and vacuum with a density  $\rho_v \simeq 0$  occupy the spatial regions  $z \geq +\frac{d}{2}$  and  $z \leq +\frac{d}{2}$ , respectively, the solution of a chameleon field in the spatial region  $z \leq +\frac{d}{2}$  is

$$\phi(z) = \phi_d \left( 1 + \frac{n+2}{\sqrt{2n(n+1)}} \frac{1}{\lambda_d} \left( \frac{d}{2} - z \right) \right)^{\frac{2}{n+2}}, \quad (20)$$

where  $\phi_d = \phi(+d/2) = \frac{n+1}{n} \phi_m$ .

The solutions Eq.(16) and Eq.(20) agree well with the solution, obtained in [3]. Of course, the solution for a chameleon field in the spatial region above a mirror, i.e.  $z \geq -\frac{d}{2}$ , is a meaningful from the point of view of the contribution of a chameleon field to the transition frequencies of the quantum gravitational states of ultra-cold neutrons [3]. Following [3] we define a contribution of a chameleon field to the gravitational potential of the Earth coupled to ultra-cold neutrons above a mirror as

$$\Phi(z) = mg \left( \frac{d}{2} + z \right) + \beta \frac{m}{M} \phi(z), \quad (21)$$

where  $m$  and  $g$  are the neutron mass and the gravitational acceleration, respectively. In comparison with [3] the mirror is shifts down to  $z = -\frac{d}{2}$ .

The calculation of the contribution of a chameleon field, for example, to the transition frequencies of the quantum gravitational states of ultra-cold neutrons is related to the calculation of the matrix elements  $\langle p | \phi(z) \rangle$ , where  $|p\rangle$  is a low-lying quantum state of ultra-cold neutrons in the gravitational field of the Earth [6, 7]. Due the Airy functions, describing the gravitational states of ultra-cold neutrons, the main contribution to matrix elements  $\langle p | \phi(z) | p \rangle$  comes from the region around  $(\frac{d}{2} + z) \sim \ell_0 = (2m^2g)^{-1/3} = 5.9 \mu\text{m}$ , which is a natural scale for neutron quantum states in the gravitational field of the Earth [6, 7]. In this case in the strong coupling limit  $\beta \geq 10^5$  due to a relation Eq.(19) and the experimental density of a mirror  $\rho_m = 2.51 \text{ g/cm}^3$  [5] one may find that the contribution of the first term in the parentheses of Eqs.(17) and (20) can be neglected and the solutions for a chameleon field take the form

$$\phi(z) = \Lambda \left( \frac{n+2}{\sqrt{2}} \Lambda \left( \frac{d}{2} \pm z \right) \right)^{\frac{2}{n+2}}, \quad (22)$$

independent of the coupling constant  $\beta$ . This agrees well with the result, obtained in [3].

### III. THE CHAMELEON FIELD COUPLED TO ULTRA-COLD NEUTRONS BOUNCING BETWEEN TWO MIRRORS

Following the geometry used in [4] for the calculation of a contribution of a chameleon field to the Casimir force induced between two parallel plates perpendicular to the  $z$ -axis, we let ultra-cold neutrons bounce in the region  $z^2 \leq \frac{d^2}{4}$  with a density  $\rho_v$ . The regions  $z^2 \geq \frac{d^2}{4}$  are occupied by neutron mirrors with a density  $\rho_m$  such as  $\rho_m \gg \rho_v$ .

For such a geometry a chameleon field depends on the variable  $z$  only. It is described by the equation of motion

$$\frac{d^2\phi}{dz^2} = V'(\phi) - V'(\phi_{\min}). \quad (23)$$

This equation is valid for a chameleon field in three spatial regions  $z^2 \geq \frac{d^2}{4}$  and  $z^2 \leq \frac{d^2}{4}$ . At  $z = \pm \frac{d}{2}$  a chameleon field has to satisfy the standard boundary conditions

$$\begin{aligned} \phi(z) \Big|_{z \rightarrow \pm \frac{d}{2} -} &= \phi(z) \Big|_{z \rightarrow \pm \frac{d}{2} +} = \phi_d, \\ \frac{d\phi(z)}{dz} \Big|_{z \rightarrow \pm \frac{d}{2} -} &= \frac{d\phi(z)}{dz} \Big|_{z \rightarrow \pm \frac{d}{2} +}. \end{aligned} \quad (24)$$

In the spatial regions  $z^2 \geq \frac{d^2}{4}$  after the first integration Eq.(23) reduces to the first order differential equation (see Eq.(11))

$$\begin{aligned} \frac{1}{2} \left( \frac{d\phi}{dz} \right)^2 &= V(\phi) - V(\phi_m) - (\phi - \phi_m) V'(\phi_m) = \\ &= \frac{\Lambda^{4+n}}{\phi^n} \left( 1 - (n+1) \frac{\phi^n}{\phi_m^n} + n \frac{\phi^{n+1}}{\phi_m^{n+1}} \right). \end{aligned} \quad (25)$$

where the integration constants are defined from the asymptotic conditions at  $z \rightarrow \pm\infty$ . Below we use Eq.(25) only for the boundary conditions as we have done in the case of one mirror in section II.

Before the integration of the equation of motion for a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$  we have to accept that a chameleon field has a minimum at  $\phi_v$ , which is caused by the properties of the effective potential  $V_{\text{eff}}(\phi)$  but not a spatial region of a location of a chameleon field. Since the region of a localisation of a chameleon field  $z^2 \leq \frac{d^2}{4}$  is finite, such a minimum can be never reached inside.

After the first integration the equation of motion for a chameleon field in  $z^2 \leq \frac{d^2}{4}$  reduces to the first order differential equation

$$\frac{1}{2} \left( \frac{d\phi}{dz} \right)^2 = \bar{C} + V(\phi) - \phi V'(\phi_v), \quad (26)$$

where  $\bar{C}$  is a constant of integration. Since the asymptotic regions  $z \rightarrow \pm\infty$  are not reachable for a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$ , a constant  $\bar{C}$  should be determined at other auxiliary conditions [4]. In our

approach it is convenient to redefine the constant  $\bar{C}$  as follows  $\bar{C} = -\Lambda^4(1-C)$ . Taking into account that  $\rho_v \simeq 0$  and  $\phi_v \rightarrow \infty$  we transcribe Eq.(26) into the form

$$\frac{1}{2} \left( \frac{d\phi}{dz} \right)^2 = \Lambda^4 \left( C + \frac{\Lambda^n}{\phi^n} \right). \quad (27)$$

Since the second derivative of a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$  is always negative

$$\frac{d^2\phi}{dz^2} = V'(\phi) = -\frac{n\Lambda^{4+n}}{\phi^{n+1}}, \quad (28)$$

a chameleon field can never reach a minimum between two mirrors. This agrees well with the assumption that a chameleon field reaches a minimum only at  $z \rightarrow \pm\infty$ . Then, due to a symmetry of the spatial regions invariant under a transformation  $z \leftrightarrow -z$  a chameleon field as a scalar field should satisfy the constraint  $\phi(z) = \phi(-z)$ . Below following [3] we solve a problem of a chameleon field between two mirrors assuming that a derivative of a chameleon field at  $z = 0$  is continuous.

#### A. The chameleon field in the spatial region $z^2 \leq \frac{d^2}{4}$ with continuous derivative at $z = 0$

Due to a symmetry of a spatial region  $z^2 \leq \frac{d^2}{4}$  a requirement of a continuity of a derivative of a chameleon field assumes that

$$\left. \frac{d\phi}{dz} \right|_{z=0} = 0. \quad (29)$$

Setting  $z = 0$  in Eq.(27) and using the constraint Eq.(29) we obtain the constant  $C$

$$C = -\frac{\Lambda^{4+n}}{\phi_0}, \quad (30)$$

where  $\phi_0 = \phi(0)$ . As a result Eq.(27) takes the form

$$\frac{1}{2} \left( \frac{d\phi}{dz} \right)^2 = \Lambda^4 \left( \frac{\Lambda^n}{\phi^n} - \frac{\Lambda^n}{\phi_0^n} \right). \quad (31)$$

A solution of this equation can be written in the following standard form

$$\sqrt{2}\Lambda z = \pm \frac{1}{\Lambda^{\frac{n+2}{2}}} \int_{\phi_0}^{\phi} \frac{\varphi^{\frac{n}{2}} d\varphi}{\sqrt{1 - \frac{\varphi^n}{\phi_0^n}}}. \quad (32)$$

For the subsequent analysis of the solution of Eq.(32) we propose to make the following changes of variables. First, making a change of variables

$$t = \sqrt{1 - \frac{\varphi^n}{\phi_0^n}} \quad (33)$$

we transcribe Eq.(32) into the form

$$\sqrt{\frac{n}{2(n+1)}} \frac{z}{\lambda_0} = \mp \int_0^{t(\phi)} \frac{dt}{(1-t^2)^{\frac{n-2}{2n}}}, \quad (34)$$

where  $1/\lambda_0 = \Lambda\sqrt{n(n+1)}(\Lambda/\phi_0)^{(n+2)/2}$  and  $t(\phi) = \sqrt{1 - \phi^n/\phi_0^n}$ .

The integral in the r.h.s. of Eq.(34) can be represented in a simpler form. After the change  $t = \sin \psi$  for  $0 \leq \psi \leq \frac{\pi}{2}$  we arrive at the equation

$$\sqrt{\frac{n}{2(n+1)}} \frac{z}{\lambda_0} = \mp \int_0^{\psi(\phi)} \cos^{\frac{2}{n}} \psi d\psi, \quad (35)$$

where  $\psi(\phi) = \arcsin \sqrt{1 - \phi^n/\phi_0^n}$ .

The integral in the r.h.s. of Eq.(34) can be represented in terms of the incomplete Beta function [8]. For this aim we use a relation

$$\int_0^{t(\phi)} \frac{dt}{(1-t^2)^{\frac{n-2}{2n}}} = \frac{1}{2} B\left(t^2(\phi); \frac{1}{2}, \frac{n+2}{2n}\right). \quad (36)$$

In terms of the incomplete Beta function Eq.(34) reads

$$\sqrt{\frac{n}{2(n+1)}} \frac{z}{\lambda_0} = \mp \frac{1}{2} B\left(t^2(\phi); \frac{1}{2}, \frac{n+2}{2n}\right). \quad (37)$$

From the boundary conditions for a chameleon field at  $z = \mp \frac{d}{2}$  we obtain a relation between  $\phi_d$  and  $\phi_0$

$$\sqrt{\frac{n}{2(n+1)}} \frac{d}{\lambda_0} = B\left(1 - \frac{\phi_d^n}{\phi_0^n}; \frac{1}{2}, \frac{n+2}{2n}\right). \quad (38)$$

Using the boundary conditions for the derivatives of a chameleon field we relate  $\phi_d$  to  $\phi_m$  and  $\phi_0$  as follows

$$\phi_d = \frac{n+1}{n} \phi_m - \frac{\phi_m^{n+1}}{\phi_0^n}. \quad (39)$$

Suppose that  $\phi_0$  does not depend on  $\beta$  in the strong coupling limit  $\beta \gg 1$ . We confirm such a property of  $\phi_0$  below for the exact solution obtained for  $n = 2$ . We also show that for the experimental density  $\rho_m = 2.51 \text{ g/cm}^3$  of a mirror and  $d = 25.5 \mu\text{m}$  [5] the independence of  $\phi_0$  of the coupling constant  $\beta$  starts for  $\beta \geq 10^5$ . Since in the strong coupling limit  $\phi_m \gg \phi_m^{n+1}/\phi_0^n$ ,  $\phi_d$  is related to  $\phi_m$  as

$$\phi_d = \frac{n+1}{n} \phi_m. \quad (40)$$

Substituting Eq.(40) into Eq.(38) and taking into account that in the strong coupling limit  $\phi_0 \gg \phi_d$  we arrive at the equation

$$\sqrt{\frac{n}{2(n+1)}} \frac{d}{\lambda_0} = \sqrt{\pi} \frac{\Gamma\left(\frac{n+2}{2n}\right)}{\Gamma\left(\frac{n+1}{n}\right)}. \quad (41)$$

For the derivation of Eq.(41) we have used a relation [8]

$$B\left(1; \frac{1}{2}, \frac{n+2}{2n}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+2}{2n}\right)}{\Gamma\left(\frac{n+1}{n}\right)}, \quad (42)$$

where  $\Gamma(a)$  is Euler's Gamma function and  $\Gamma(1/2) = \sqrt{\pi}$  [8]. Solving Eq.(41) with respect to  $\phi_0$  we obtain  $\phi_0$  as a function of  $\Lambda$ ,  $d$  and  $n$  only

$$\phi_0 = \Lambda \left( \frac{n}{\sqrt{2\pi}} \frac{\Gamma\left(\frac{n+1}{n}\right)}{\Gamma\left(\frac{n+2}{2n}\right)} \Lambda d \right)^{\frac{2}{n+2}}. \quad (43)$$

Thus, we may assert that in the spatial region  $z^2 \leq \frac{d^2}{4}$  in the strong coupling limit  $\beta \geq 10^5$ , being valid for the experimental density of a mirror  $\rho_m = 2.51 \text{ g/cm}^3$  [5] and  $d = 25.5 \mu\text{m}$ , any solution of a chameleon field with a continuous derivative at  $z = 0$  should depend on two parameters  $\phi_d = \frac{n+1}{n} \phi_m$  and  $\phi_0$ , determined by Eq.(4) for  $\rho = \rho_m$  and Eq.(43), respectively. This assertion we prove below for the exactly solvable case  $n = 2$ .

**B. The chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$  with continuous derivative at  $z = 0$ . Exact solution for  $n = 2$**

As it is seen from Eq.(32) for  $n = 2$  the integral over  $\varphi$  can be calculated in terms of elementary functions. Setting  $n = 2$  and integrating over  $\varphi$  we obtain

$$\sqrt{2}\Lambda z = \mp \frac{\phi_0^2}{\Lambda^2} \sqrt{1 - \frac{\phi^2}{\phi_0^2}}. \quad (44)$$

This defines  $\phi$  as a function of  $z$

$$\phi(z) = \phi_0 \left( 1 - \frac{1}{3} \frac{z^2}{\lambda_0^2} \right)^{1/2}, \quad (45)$$

where  $1/\lambda_0 = \Lambda\sqrt{6}(\Lambda/\phi_0)^2$ . At the next step we have to express the parameter  $\phi_0$  in terms of  $\phi_d = \phi(\pm d/2)$ . Setting  $z = \pm \frac{d}{2}$  we obtain

$$\frac{\phi_d^2}{\phi_0^2} = 1 - \frac{d^2}{12\lambda_d^2} \frac{\phi_d^4}{\phi_0^4}, \quad (46)$$

where  $1/\lambda_d = \Lambda\sqrt{6}(\Lambda/\phi_d)^2$ . The solution of this algebraical equation with respect to  $\phi_0^2$  is

$$\frac{\phi_d^2}{\phi_0^2} = \frac{6\lambda_d^2}{d^2} \left( \sqrt{1 + \frac{d^2}{3\lambda_d^2}} - 1 \right) = \frac{2}{1 + \sqrt{1 + \frac{d^2}{3\lambda_d^2}}}. \quad (47)$$

Thus, the solution for a chameleon field, defined for  $n = 2$ , is

$$\phi(z) = \phi_d \sqrt{1 + \frac{1}{3} \frac{1}{\Lambda_d^2} \left( \frac{d^2}{4} - z^2 \right)}, \quad (48)$$

where  $\Lambda_d$  is defined by

$$\Lambda_d = \lambda_d \sqrt{\frac{1 + \sqrt{1 + \frac{d^2}{3\lambda_d^2}}}{2}} = \lambda_d \frac{\phi_0}{\phi_d}. \quad (49)$$

Using the relation

$$\begin{aligned} \frac{1}{3} \frac{\phi_d^4}{\Lambda_d^2} \left( 1 + \frac{1}{12} \frac{d^2}{\Lambda_d^2} \right) &= \frac{1}{3} \frac{\phi_d^4}{\lambda_d^2} \frac{2}{1 + \sqrt{1 + \frac{d^2}{3\lambda_d^2}}} \\ &\times \left( 1 + \frac{d^2}{12\lambda_d^2} \frac{2}{1 + \sqrt{1 + \frac{d^2}{3\lambda_d^2}}} \right) = \\ &= \frac{1}{3} \frac{\phi_d^4}{\lambda_d^2} \frac{2}{\left( 1 + \sqrt{1 + \frac{d^2}{3\lambda_d^2}} \right)^2} \left( 1 + \sqrt{1 + \frac{d^2}{3\lambda_d^2}} + \frac{d^2}{6\lambda_d^2} \right) = \\ &= \frac{1}{3} \frac{\phi_d^4}{\lambda_d^2} = \frac{1}{3} \times \phi_d^4 \times 6 \times \frac{\Lambda^6}{\phi_d^4} = 2 \times \Lambda^6 \end{aligned} \quad (50)$$

one can show that the solution Eq.(48) satisfies Eq.(28) at  $n = 2$ .

From the boundary conditions for the first derivatives of a chameleon field we may define the parameter  $\phi_d$  in terms of  $\phi_m$

$$\phi_d = \frac{3}{2} \phi_m - \frac{1}{2} \frac{\phi_m^3}{\phi_0^2}. \quad (51)$$

If  $\phi_0$  does not depend on the coupling constant  $\beta$  the second term in the r.h.s. of Eq.(51), taken in the strong coupling limit  $\beta \geq 10^5$ , should be smaller compared with the first one. Neglecting the contribution of the second term we obtain that in the strong coupling limit  $\phi_d$  can be approximated by  $\phi_d = \frac{3}{2} \phi_m$ . This agrees well with Eq.(40) at  $n = 2$ . Since for  $n = 2$  the problem is exactly solvable, we may check such a supposition by using the exact relation between  $\phi_d$  and  $\phi_0$ , given by Eq.(47), and analysing the solution of Eq.(51) numerically.

For this aim we transcribe Eq.(51) into the form

$$2 \frac{\phi_d^3}{\phi_m^3} - 3 \frac{\phi_d^2}{\phi_m^2} + \frac{2}{1 + \sqrt{1 + \frac{d^2}{3\lambda_d^2}}} = 0, \quad (52)$$

where we have used Eq.(47). Denoting  $X = \phi_d/\phi_m$  we rewrite Eq.(52) as follows

$$f(X) = X^3 - \frac{3}{2} X^2 + \frac{1}{1 + \sqrt{1 + \frac{d^2}{3\lambda_m^2} \frac{1}{X^4}}} = 0, \quad (53)$$

where  $\lambda_m$  is defined by

$$\frac{1}{\lambda_m} = \Lambda\sqrt{6} \left( \frac{\Lambda}{\phi_m} \right)^2 = \Lambda\sqrt{6} \left( \frac{\beta\rho_m}{2M\Lambda^3} \right)^{2/3}. \quad (54)$$

For numerical analysis we use  $\rho_m = 2.51 \text{ g/cm}^3$  and  $d = 25.5 \mu\text{m}$  [5]. One can show that the function  $f(X)$  has a real root, which for  $\beta \geq 10^5$  is practically equal to  $X = 3/2$ . This confirms our supposition that the parameter  $\phi_0$  does not depend on the coupling constant  $\beta$  in the strong coupling limit.

In the strong coupling limit  $\beta \geq 10^5$ , where  $\phi_d = (3/2)\phi_m$ , the ratio  $\phi_d^2/\phi_0^2$  is given by

$$\frac{\phi_d^2}{\phi_0^2} = 2\sqrt{3} \frac{\lambda_d}{d} = \frac{2\sqrt{3}}{d} \frac{1}{\Lambda\sqrt{6}} \frac{\phi_d^2}{\Lambda^2} = \frac{\sqrt{2}}{\Lambda d} \frac{\phi_d^2}{\Lambda^2}. \quad (55)$$

Since  $\phi_d^2$  is cancelled, we obtain  $\phi_0$  as a function of  $\Lambda$  and  $d$

$$\phi_0 = \frac{\Lambda}{2^{1/4}} \sqrt{\Lambda d}. \quad (56)$$

This confirms our assertion about the independence of  $\phi_0$  of the coupling constant  $\beta$  in the strong coupling limit and, correspondingly, Eq.(43) for  $n = 2$ . The scale  $\lambda_0$  is equal to

$$\lambda_0 = \frac{1}{\Lambda\sqrt{6}} \frac{\phi_0^2}{\Lambda^2} = \frac{1}{\Lambda\sqrt{6}} \frac{1}{\Lambda^2} \frac{1}{\sqrt{2}} \Lambda^2 (\Lambda d) = \frac{d}{2\sqrt{3}}. \quad (57)$$

Thus, in the strong coupling limit the solution Eq.(48) can be transcribed into the form

$$\begin{aligned} \phi(z) &= \frac{\Lambda}{2^{1/4}} \sqrt{\Lambda d} \left(1 - 4 \frac{z^2}{d^2}\right)^{1/2} = \\ &= 2^{3/4} \sqrt{\frac{\Lambda}{d}} \left(\Lambda^2 \left(\frac{d^2}{4} - z^2\right)\right)^{1/2}. \end{aligned} \quad (58)$$

In the vicinity of  $z \simeq \pm \frac{d}{2}$ , where the contribution of the mirror, localised at  $z = \pm \frac{d}{2}$ , can be neglected and the problem under consideration reduces to the problem of a chameleon field coupled to ultra-cold neutrons, bouncing in the gravitational field of the Earth above a mirror, the solution Eq.(58) takes the form

$$\phi(z) = 2^{3/4} \Lambda \left(\Lambda \left(\frac{d}{2} \pm z\right)\right)^{1/2}. \quad (59)$$

It agrees well with the solution Eq.(22), taken in the strong coupling limit.

As we have shown the exact solution of a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$ , carried out for  $n = 2$ , confirms fully our suppositions, concerning a dependence of the parameters  $\phi_d$  and  $\phi_0$  on the coupling constant  $\beta$  in the strong coupling limit.

The solution Eq.(58) can be applied to the analysis of the contribution of a chameleon field to the transition frequencies of the quantum gravitational states of ultra-cold neutrons bouncing in the gravitational field between two mirrors. Following [3], the gravitational potential, corrected by the contribution of a chameleon field, takes the form Eq.(21).

As we have already pointed out that due to the Airy wave functions the solution of a chameleon field is localised in the vicinity of  $(\frac{d}{2} + z) \sim \ell_0 = (2m^2g)^{-1/3} =$

$5.9 \mu\text{m}$ , which is a natural scale for neutron quantum states in the gravitational field of the Earth [6, 7]. Indeed, since a contribution of a chameleon field to the gravitational potential of the interaction with ultra-cold neutrons to the matrix elements  $\langle p|\phi|p\rangle$  for low-lying gravitational states  $|p\rangle = |1\rangle$  and  $|3\rangle$  is localised around  $(\frac{d}{2} + z) \sim \ell_0$  due to the Airy functions [6, 7], the solution Eq.(58), obtained in the strong coupling limit  $\beta \geq 10^5$ , should be valid around  $(\frac{d}{2} + z) \sim \ell_0$ . This agrees also well with [3].

### C. The chameleon field in the spatial region $z^2 \leq \frac{d^2}{4}$ with continuous derivative at $z = 0$ . Approximate solution of non-linear equation for arbitrary $n$

Now we may proceed to solving Eq.(34) for arbitrary  $n$ . The only approximation, which may lead to an analytical solution of the problem, is  $t \ll 1$ . This allows to get an agreement with the exact solution at  $n = 2$ . Such an approximation gives us the so-called *low-contrast solution* [2]. Neglecting in Eq.(34) the term  $t^2$  with respect to a unity and integrating over  $t$  we arrive at the equation

$$\sqrt{2} \Lambda z = \mp \frac{2}{n} \left(\frac{\phi_0}{\Lambda}\right)^{\frac{n+2}{2}} \sqrt{1 - \frac{\phi^n}{\phi_0^n}}. \quad (60)$$

Solving this equation with respect to  $\phi$  we obtain a solution for a chameleon field

$$\phi(z) = \phi_0 \left(1 - \frac{n}{2(n+1)} \frac{z^2}{\lambda_0^2}\right)^{1/n}, \quad (61)$$

where  $1/\lambda_0 = \Lambda \sqrt{n(n+1)(\Lambda/\phi_0)^{n+2}}$ . At  $n = 2$  Eq.(61) reduces to Eq.(45).

Now we have to replace the parameter  $\phi_0$  by  $\phi_d$ . Setting  $z = \pm \frac{d}{2}$  we obtain

$$\frac{\phi_d^n}{\phi_0^n} = 1 - \frac{n}{8(n+1)} \frac{d^2}{\lambda_d^2} \frac{\phi_d^{n+2}}{\phi_0^{n+2}}, \quad (62)$$

where  $1/\lambda_d = \Lambda \sqrt{n(n+1)(\Lambda/\phi_d)^{n+2}}$ . For the subsequent calculations it is convenient to rewrite Eq.(62) as follows

$$\frac{\phi_d^n}{\phi_0^n} = \frac{1}{1 + \frac{n}{8(n+1)} \frac{d^2}{\lambda_d^2} \frac{\phi_d^{n+2}}{\phi_0^{n+2}}}. \quad (63)$$

Using Eq.(63) we transcribe Eq.(61) into the form

$$\begin{aligned}
\phi(z) &= \phi_0 \left( 1 - \frac{n}{2(n+1)} \frac{z^2}{\lambda_0^2} \right)^{1/n} = \\
&= \phi_0 \left( 1 - \frac{n}{2(n+1)} \frac{z^2}{\lambda_d^2} \frac{\phi_d^{n+2}}{\phi_0^{n+2}} \right)^{1/n} = \\
&= \phi_0 \left( 1 - \frac{n}{2(n+1)} \frac{z^2}{\lambda_d^2} \frac{\phi_d^2}{\phi_0^2} \frac{1}{1 + \frac{n}{8(n+1)} \frac{d^2}{\lambda_d^2} \frac{\phi_d^2}{\phi_0^2}} \right)^{1/n} = \\
&= \frac{\phi_0}{\left( 1 + \frac{n}{8(n+1)} \frac{d^2}{\lambda_d^2} \frac{\phi_d^2}{\phi_0^2} \right)^{1/n}} \\
&\times \left( 1 + \frac{n}{2(n+1)} \frac{1}{\lambda_d^2} \left( \frac{d^2}{4} - z^2 \right) \frac{\phi_d^2}{\phi_0^2} \right)^{1/n} = \\
&= \phi_d \left( 1 + \frac{n}{2(n+1)} \frac{1}{\Lambda_d^2} \left( \frac{d^2}{4} - z^2 \right) \right)^{1/n}, \quad (64)
\end{aligned}$$

where we have denoted  $\Lambda_d = \lambda_d(\phi_0/\phi_d)$ . Thus, an approximate solution for a chameleon field for an arbitrary  $n$  is

$$\phi(z) = \phi_d \left( 1 + \frac{n}{2(n+1)} \frac{1}{\Lambda_d^2} \left( \frac{d^2}{4} - z^2 \right) \right)^{1/n}. \quad (65)$$

For  $n = 2$  the function Eq.(61) reduces to the function Eq.(48).

From Eq.(62) in the strong coupling limit we define  $\phi_0$  by the expression

$$\phi_0 = \Lambda \left( \frac{n}{2\sqrt{2}} \Lambda d \right)^{\frac{2}{n+2}}. \quad (66)$$

For  $1 \leq n \leq 10$  such an expression reproduces the exact result Eq.(43) with an accuracy better than 6.3%. Thus, we may assert that the property of  $\phi_0$  to be independent of the coupling constant  $\beta$  in the strong coupling limit is a general property, which does not depend on the approximation, but an exact dependence of  $\phi_0$  on the parameters  $\Lambda$ ,  $d$  and  $n$  depends, of course, on it.

In order to show that the function Eq.(65) can be used as a solution for a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$  and satisfies the equation of motion for a chameleon field, we have to derive the equation of motion Eq.(28) at the same approximation, which we have used for the derivation of Eq.(65). For this aim we have to derive the equation of motion for  $t(\phi)$ . The first derivative of  $\phi$  is equal to

$$\frac{d\phi}{dz} = -\frac{2}{n} \phi_0 t(1-t^2)^{\frac{1}{n}} - 1 \frac{dt}{dz} \quad (67)$$

and the second one is

$$\begin{aligned}
\frac{d^2\phi}{dz^2} &= -\frac{2}{n} \phi_0 t(1-t^2)^{\frac{1}{n}} - 1 \frac{d^2t}{dz^2} \\
&- \frac{2}{n} \phi_0 (1-t^2)^{\frac{1}{n}} - 1 \left( \frac{dt}{dz} \right)^2 \\
&- \frac{2}{n} \frac{2(n-1)}{n} \phi_0 (1-t^2)^{\frac{1}{n}} - 2 \left( \frac{dt}{dz} \right)^2. \quad (68)
\end{aligned}$$

The r.h.s. of Eq.(28) can be transcribed into the form

$$-\frac{n\Lambda^{n+4}}{\phi^{n+1}} = -\frac{n\Lambda^{n+4}}{\phi_0^{n+1}} \frac{1}{(1-t^2)^{\frac{n+1}{n}}}. \quad (69)$$

Thus, the equation of motion Eq.(28), rewritten for  $t(\phi)$ , takes the form

$$\begin{aligned}
t \frac{d^2t}{dz^2} + \left( \frac{dt}{dz} \right)^2 + \frac{2(n-1)}{n} \frac{t^2}{1-t^2} \left( \frac{dt}{dz} \right)^2 &= \\
= \frac{n^2 \Lambda^{n+4}}{2 \phi_0^{n+1}} \frac{1}{(1-t^2)^{\frac{2}{n}}}. \quad (70)
\end{aligned}$$

Neglecting the contributions of the terms of order  $O(t^2)$  we reduce Eq.(70) to the form

$$t \frac{d^2t}{dz^2} + \left( \frac{dt}{dz} \right)^2 = \frac{n^2 \Lambda^{n+4}}{2 \phi_0^{n+2}}. \quad (71)$$

In our approximation the solution of Eq.(34) is

$$t = \pm \frac{n}{\sqrt{2}} \frac{\Lambda^{\frac{n+4}{2}}}{\phi_0^{\frac{n+2}{2}}} z. \quad (72)$$

Substituting Eq.(72) into Eq.(71) we satisfy the approximate equation of motion. This means that we may use the solution Eq.(65) for a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$ .

Since we are interested in the solution for a chameleon field in the strong coupling limit, using the relations  $\phi_d = \frac{n+1}{n} \phi_m$  and Eq.(66) we reduce the solution Eq.(65) to the form

$$\begin{aligned}
\phi(z) &= \Lambda 2^{\frac{n-4}{2n(n+2)}} \frac{2}{n^{\frac{2}{n+2}}} (\Lambda d)^{-\frac{4}{n(n+2)}} \\
&\times \left( \Lambda^2 \left( \frac{d^2}{4} - z^2 \right) \right)^{\frac{1}{n}}. \quad (73)
\end{aligned}$$

We would like to remind that the solution Eq.(73) is valid for  $\beta \geq 10^5$  if the mirror density is equal to  $\rho_m = 2.51 \text{ g/cm}^3$  and  $d = 25.5 \text{ g/cm}^3$  [5]. For  $n = 2$  it coincides with Eq.(58). In the vicinity of the mirrors  $z = \mp \frac{d}{2}$ , the solution Eq.(73) takes the form

$$\phi(z) = \Lambda 2^{\frac{n-4}{2n(n+2)}} \frac{2}{n^{\frac{2}{n+2}}} (\Lambda d)^{\frac{n-2}{n+2}} \left( \Lambda \left( \frac{d}{2} \pm z \right) \right)^{\frac{1}{n}}. \quad (74)$$

One may see that the solution Eq.(74) reproduces the solution for a chameleon field above (below) a mirror (see Eq.(22)) only for  $n = 2$ .

We would like to note that it is obvious that even if the solution for a chameleon field with  $n = 2$  and arbitrary coupling constant  $\beta$  does not reproduce in the vicinity of a mirror the solution for a chameleon field above one mirror. This means that fact that the solution Eq.(73) does not reproduce in the vicinity of a mirror the solution for a chameleon field above a mirror (see Eq.(22) should not be evaluated as a strong argument against to use such a solution for the problem under consideration.

That is why in section IV we apply the solution Eq.(73) to the estimate of the coupling constant  $\beta$  from the contribution of a chameleon field to the transition frequencies of quantum gravitational states of ultra-cold neutrons, bouncing in the gravitational field of the Earth between two mirrors [5].

**D. The chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$  with continuous derivative at  $z = 0$ . Solution of the linearised equation of motion for arbitrary  $n$**

In order to complete the analysis of solutions for a chameleon field, possessing a continuous derivative at  $z = 0$ , we have to consider the linearised equations of motion. In the spatial region  $z^2 \leq \frac{d^2}{4}$  the linearised Eq.(28) takes the form

$$\frac{d^2\phi}{dz^2} = -n\frac{\Lambda^{4+n}}{\phi_0^{n+1}} - n(n+1)\frac{\Lambda^{4+n}}{\phi_0^{n+2}}(\phi_0 - \phi). \quad (75)$$

Denoting  $\phi_0 - \phi = \varphi$  we define the following linearised equation for a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$

$$\frac{d^2\varphi}{dz^2} - \frac{\varphi}{\lambda_0^2} = \frac{n\Lambda^{4+n}}{\phi_0^{n+1}}, \quad (76)$$

where  $1/\lambda_0^2 = n(n+1)\Lambda^{n+4}/\phi_0^{n+2}$ . As  $\phi(z) = \phi(-z)$  the solution of Eq.(76) is

$$\varphi(z) = -\lambda_0^2 \frac{n\Lambda^{4+n}}{\phi_0^{n+1}} + A \cosh\left(\frac{z}{\lambda_0}\right), \quad (77)$$

where  $A$  is a constant of integration. Since  $\varphi(0) = 0$ , we get

$$\begin{aligned} \varphi(z) &= -\lambda_0^2 \frac{\Lambda^{4+n}}{\phi_0^{n+1}} \left(1 - \cosh\left(\frac{z}{\lambda_0}\right)\right) = \lambda_0^2 \frac{2n\Lambda^{4+n}}{\phi_0^{n+1}} \\ &\times \sinh^2\left(\frac{z}{2\lambda_0}\right) = \phi_0 \frac{2}{n+1} \sinh^2\left(\frac{z}{2\lambda_0}\right). \end{aligned} \quad (78)$$

This gives a chameleon field  $\phi(z)$  equal to

$$\phi(z) = \phi_0 \left(1 - \frac{2}{n+1} \sinh^2\left(\frac{z}{2\lambda_0}\right)\right). \quad (79)$$

This solution agrees well with that obtained in [4] (see Eq.(20) of [4] at  $V'_b = 0$ ).

Since we are interesting in the solution for a chameleon field in the strong coupling limit, the solution Eq.(79) may be valid in the strong coupling limit if it satisfies the following constraint. Indeed, in the strong coupling limit  $\phi_d$  is proportional to  $\phi_m$ , i.e.  $\phi_d = \frac{n+1}{n} \phi_m$ , and commensurable with zero. This implies that the solution Eq.(79) should vanish at  $z = \pm \frac{d}{2}$  (see Eq.(58) and Eq.(73)). Setting  $z = \pm \frac{d}{2}$  we obtain a constraint on  $\phi_0$

$$\phi_0 = \Lambda \left( \frac{\sqrt{n(n+1)}}{4} \frac{\Lambda d}{\ell n \left( \sqrt{\frac{n+1}{2}} + \sqrt{\frac{n+3}{2}} \right)} \right)^{\frac{2}{n+2}}. \quad (80)$$

One can show that for  $1 \leq n \leq 10$  the parameter  $\phi_0$ , defined by Eq.(80), fits the exact expression Eq.(43) with an accuracy better than 8.7%.

The solution Eq.(80) with  $\phi_0$ , defined by Eq.(80), can be also applied to the estimate of the coupling constant  $\beta$  from the contribution of a chameleon field to the transition frequencies of quantum gravitational states of ultra-cold neutrons, bouncing in the gravitational field of the Earth between two mirrors [5].

#### IV. NUMERICAL SOLUTION OF THE PROBLEM

In this section, we propose a numerical solution of Eq.(34) for a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$  between two mirrors with a density  $\rho_m = 2.51 \text{ g/cm}^3$  separated by the distance  $d = 25.5 \mu\text{m}$  [5]. First of all we would like note that due to a symmetry of the spatial region a chameleon field as a scalar field is symmetric with respect to a transformation  $z \longleftrightarrow -z$ , i.e.  $\phi(z) = \phi(-z)$ . For a numerical solution we transcribe Eq.(34) into the form

$$\sqrt{\frac{n}{2(n+1)}} \frac{|z|}{\lambda_0} = \int_0^{t(\phi)} \frac{dt}{(1-t^2)^{\frac{n-2}{2n}}}, \quad (81)$$

where  $1/\lambda_0 = \Lambda \sqrt{n(n+1)} (\Lambda/\phi_0)^{\frac{n+2}{2}}$ ,  $\phi_0$  is given by Eq.(43) and  $|z|$  is the absolute value of  $z$ . As a result we define a chameleon field as a function of  $|z|$ , i.e.  $\phi(z) = f(|z|)$ . Such a non-analytical dependence of a chameleon field on a spatial variable does not prevent it to satisfy the equation of motion Eq.(28). Indeed, the first derivative of a chameleon field with respect to  $z$  is equal to

$$\frac{d\phi(z)}{dz} = \varepsilon(z) \frac{df(|z|)}{d|z|}, \quad (82)$$

where  $\varepsilon(z)$  is a sign function, defined by  $d|z|/dz = \varepsilon(z) = \theta(z) - \theta(-z)$  and  $\theta(\pm z)$  are the Heaviside functions [11]. For the second derivative we obtain the following expres-



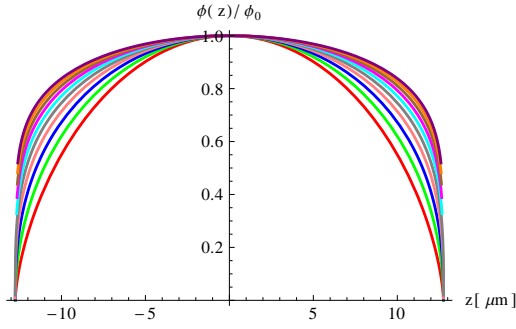


FIG. 1: The profiles of a chameleon field, calculated in the strong coupling limit as the solutions of Eq.(81) in the spatial region  $z^2 \leq \frac{d^2}{4}$  and  $n \in [1, 10]$ .

sion

$$\begin{aligned} \frac{d^2\phi(z)}{dz^2} &= 2\delta(z) \frac{df(|z|)}{d|z|} + \varepsilon^2(z) \frac{d^2f(|z|)}{d|z|^2} = \\ &= 2\delta(z) \frac{df(|z|)}{d|z|} \Big|_{z=0} + \varepsilon^2(z) \frac{d^2f(|z|)}{d|z|^2} = \frac{d^2f(|z|)}{d|z|^2}, \end{aligned} \quad (83)$$

where we have used that  $d\varepsilon(z)/dz = 2\delta(z)$  and  $\varepsilon^2(z) = 1$  [11]. Thus, due to a continuous derivative of a chameleon field at  $z = 0$  a dependence of a chameleon field on the absolute value of  $z$  does not introduce an additional terms violating the equation of motion Eq.(28).

For  $n = 1$  the integral Eq.(81) can be calculated exactly. We obtain

$$\frac{|z|}{\lambda_0} = t\sqrt{1-t^2} + \arcsin t. \quad (84)$$

This gives a chameleon field as a function of  $t(|z|/\lambda_0)$

$$\phi(z) = \phi_0 \left( 1 - t^2 \left( \frac{|z|}{\lambda_0} \right) \right). \quad (85)$$

In Fig.1 we show the profiles of a chameleon field, obtained in the spatial region  $z^2 \leq \frac{d^2}{4}$  in the strong coupling limit by a numerical solution of Eq.(81) for the experimental mirror density  $\rho_m = 2.51 \text{ g/cm}^3$  and  $d = 25.5 \mu\text{m}$  and  $n \in [1, 10]$ . The lower red line gives the solution for  $n = 1$ , the green line corresponds to the solution for  $n = 2$  and so on. In Fig.2 we give the profiles of a chameleon field in the 3D picture. In Fig.3 we compare the numerical solutions of Eq.(81) with the approximate solution Eq.(73), which we represent in the more convenient form

$$\phi(z) = \phi_0 \left( \frac{n}{2(n+1)} \frac{1}{\lambda_0^2} \left( \frac{d^2}{4} - z^2 \right) \right)^{\frac{1}{n}}, \quad (86)$$

where  $1/\lambda_0 = \Lambda \sqrt{n(n+1)} (\Lambda/\phi_0)^{\frac{n+2}{2}}$  but  $\phi_0$  is given by Eq.(66). The dashed lines describe the approximate solutions, whereas the solid lines correspond to the exact numerical solutions of Eq.(81). One can see that for  $n =$

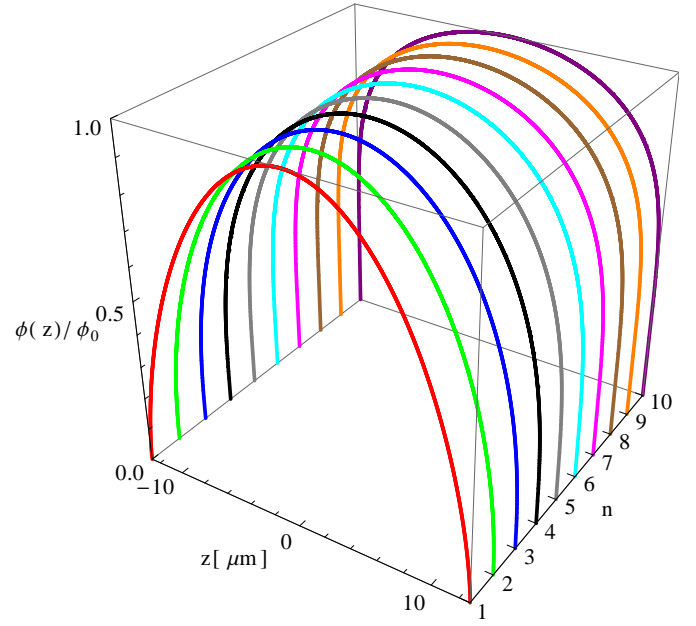


FIG. 2: The profiles of a chameleon field, calculated in the strong coupling limit as the solutions of Eq.(81) in the spatial region  $z^2 \leq \frac{d^2}{4}$  and  $n \in [1, 10]$ .

2 the dashed and solid lines coincide, since for  $n = 2$  the problem admits the analytical solution in the elementary functions. Such an agreement confirms the correctness all numerical solutions, obtained for  $n \in [1, 10]$ . Starting with  $n \geq 3$  the approximate solution is larger compared with the exact numerical solutions.

Following such a behaviour of the approximate solutions and keeping in mind that the exact solutions of a chameleon field taken in the vicinity of a mirror  $z \simeq \mp \frac{d}{2}$  should reproduce the solutions Eq.(22), for the fit of the exact numerical solutions of a chameleon field we propose the function

$$\phi(z) = \phi_0 \left( 1 - \frac{4z^2}{d^2} \right)^{\frac{2}{n+2}} \quad (87)$$

$$= \Lambda \left( \frac{n+2}{\sqrt{2}} \frac{\Lambda}{d} \left( \frac{d^2}{4} - z^2 \right) \right)^{\frac{2}{n+2}}, \quad (88)$$

where  $\phi_0$  is equal to

$$\phi_0 = \Lambda \left( \frac{n+2}{4\sqrt{2}} \Lambda d \right)^{\frac{2}{n+2}}, \quad (89)$$

The results of the fit are shown in Fig.4. One may see that Eq.(87) fits well the numerical solutions of the non-linear equation of motion of a chameleon field and can be used for the analytical description of a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$  between two mirrors with a density  $\rho_m = 2.51 \text{ g/cm}^3$  in the strong coupling limit  $\beta \geq 10^5$  and an arbitrary  $n$ . For  $n > 2$  the parameter  $\phi_0$ , given by Eq.(89) reproduces the exact  $\phi_0$ , defined by Eq.(43), with an accuracy better than 3.7%.

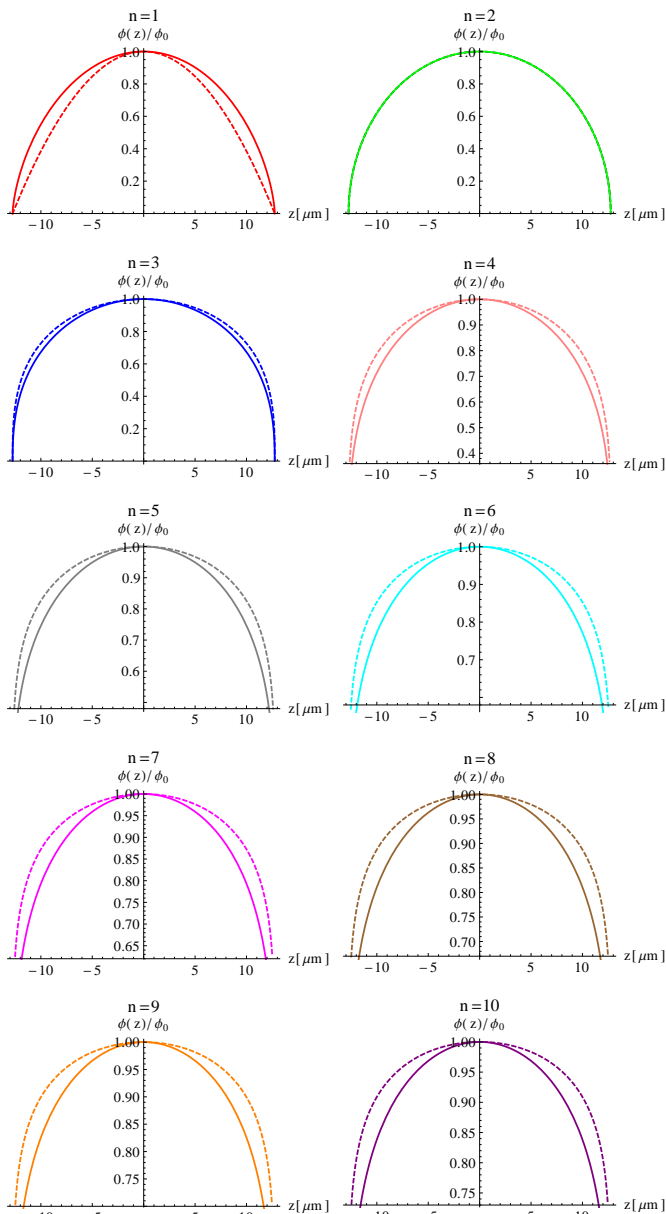


FIG. 3: The profiles (solid lines) of a chameleon field  $\phi(z)/\phi(0)$  in the spatial region  $z^2 \leq \frac{d^2}{4}$  as the solutions of Eq.(81), calculated numerically in the strong coupling limit, in comparison with the approximate solution Eq.(86) (dashed lines) for  $n \in [1, 10]$ .

## V. CONCLUSION

We have analyzed the solutions for a chameleon field, localized between two mirrors in the spatial region  $z^2 \leq \frac{d^2}{4}$ , and its influence on the transition frequencies of the quantum gravitational states of ultra-cold neutron, bouncing between two mirrors in the gravitational field of the Earth.

For the power  $n = 2$  of the potential of a chameleon field we have found the exact analytical solutions of non-

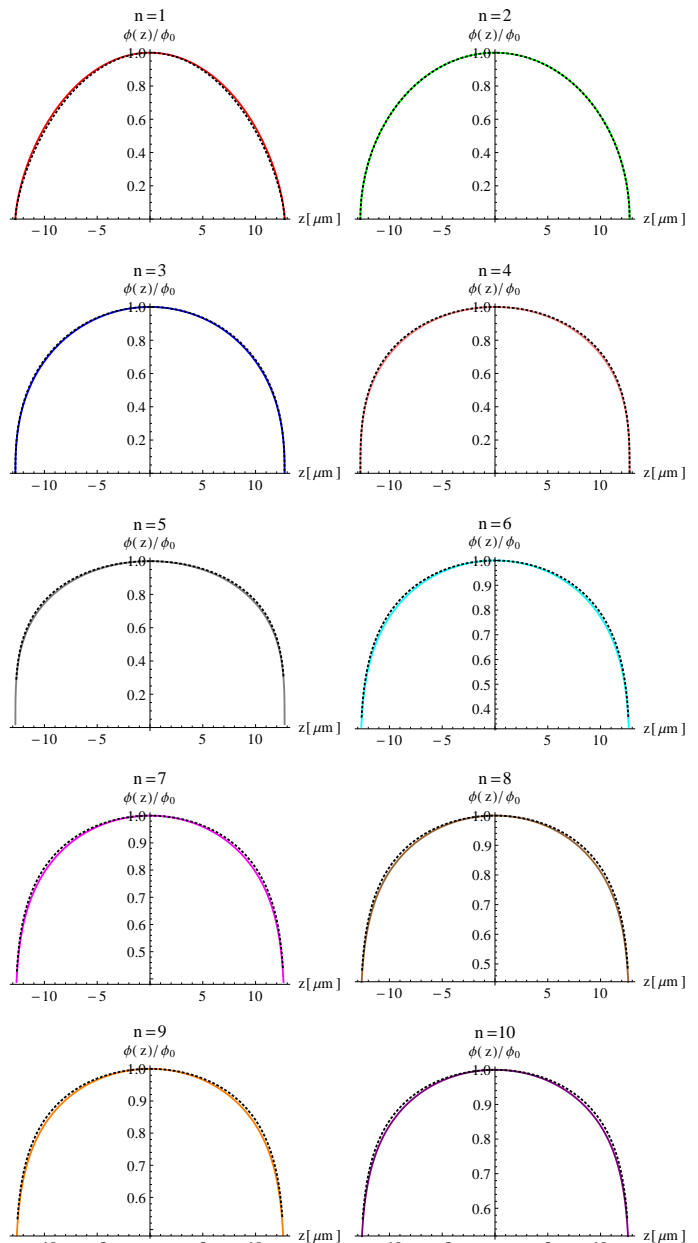


FIG. 4: The analytical fit of the numerical solutions of the non-linear equations of motion of a chameleon field  $\phi(z)/\phi(0)$  in the spatial region  $z^2 \leq \frac{d^2}{4}$ , obtained in the strong coupling limit for  $n \in [1, 10]$ .

linear equations of motion. We have shown that for the experimental density of mirrors  $\rho_m = 2.51 \text{ g/cm}^3$  and a relative distance between mirrors  $d = 25.5 \mu\text{m}$  [5] the obtained solutions obey the strong coupling regime with  $\beta \geq 10^5$ . In such a regime a chameleon field does not depend on the coupling constant. As a result the contribution of a chameleon field to the transition frequencies of ultra-cold neutrons, bouncing in the gravitational field of the Earth between two mirrors, is proportional to the coupling constant  $\beta$ .

In the strong coupling limit  $\beta > 10^5$  we have obtained the approximate solutions of non-linear and linearized equations of motion for a chameleon field and compared them with the exact numerical solutions of non-linear equations of motion and the analytical fit of the exact numerical solutions of non-linear equations of motion. The analytical fit of the exact numerical solutions is taken in the form reproducing in the vicinity of a mirror at  $z \simeq \mp \frac{d}{2}$  the solutions Eq.(22), coinciding with the solutions obtained in [3]. For  $n > 2$  the accuracy of the analytical fit of the exact numerical solutions is better than 3.82%.

Eq. 87 may be used, for example, to calculate bounds on the coupling constant  $\beta$  by comparing the transition frequency  $\omega_{ab}$  with its theoretical expectation  $\omega_{ab}^{\text{theo}}$ :

$$\omega_{ab} - \omega_{ab}^{\text{theo}} = \beta \frac{m}{M} (\langle a|\phi(z)|a\rangle - \langle b|\phi(z)|b\rangle). \quad (90)$$

Such resonant transitions between quantum states of a neutron in the gravity potential have been measured by the qBounce Collaboration [5]. So far a chameleon field has been calculated in the infinitely large spatial region above a mirror, whereas the ultra-cold neutrons bounce in the gravitational field of the Earth between two mirrors [5] with a relative distance  $d$ .

Finally we would like to mention that the non-linear Eq.(27) with  $C = 0$  admits non-linear solutions with a discontinuous derivative of a chameleon field. Such solutions are equal to the solutions Eq.(20) with a replacement  $z \rightarrow |z|$ . The first derivative of these solutions is proportional to the sign function  $\varepsilon(z)$  [11]. Since  $\varepsilon^2(z) = 1$  [11], the second derivative satisfies Eq.(28) with an additional term  $-4\pi\sigma_d\delta(z)$ , where a Dirac  $\delta$ -function  $\delta(z)$  appears as a derivative of the sign function  $\varepsilon'(z) = 2\delta(z)$  [11]. Then,  $4\pi\sigma_d$  is defined by a jump of the first derivative at  $z = 0$ , where  $\sigma_d$  has a dimension of surface density of scalar particle. The factor  $4\pi$  is introduced by analogy with electrostatic [12]. An anal-

ogy between the chameleon field theory in the thin-shell regime and the electrostatic has been drawn in [13, 14].

A deviation from Eq.(28) by the term  $-4\pi\sigma_d\delta(z)$  might imply that the Hamilton density of a chameleon field in the spatial region  $z^2 \leq \frac{d^2}{4}$  should be defined as follows

$$\mathcal{H}(z) = \frac{1}{2} \left( \frac{d\phi(z)}{dz} \right)^2 + V(\phi(z)) - 4\pi\sigma_d\delta(z)\phi(z). \quad (91)$$

The parameter  $\sigma$  can be unambiguously determined due to a self-interaction of a chameleon field. Indeed, the Hamilton density Eq.(91) defines the equation of motion Eq.(28) with the additional term  $-4\pi\sigma\delta(z)$ . Solving this equation for the regions  $-\frac{d}{2} \leq z < 0$  and  $0 < z \leq +\frac{d}{2}$ , respectively, we arrive at the solution Eq.(20) with a replacement  $z \rightarrow |z|$ . This gives  $\sigma = \sigma_d$ . When the distance between two mirrors tends to infinity, the surface density  $\sigma_d$  vanishes and we arrive at the solutions for a chameleon field above (below) one mirror.

Of course, we do not stand for the reality of such solutions due to a necessity to introduce an additional term  $-4\pi\sigma\delta(z)\phi$  to the Hamilton density, an influence of which on the properties of a chameleon field is not clear. The solutions with a discontinuous derivative might be accepted as an artifact of the solutions of Eq.(27).

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